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Quasi-socle ideals and Goto numbers of parameters

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ABSTRACT

Goto numbers $\mathrm{g}(Q) = \max\{q \in \mathbb{Z} \mid Q : \mathfrak{m}^q$ is integral over $Q\}$ for certain parameter ideals Q in a Noetherian local ring $(A, \mathfrak{m})$ with Gorenstein associated graded ring $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ are explored. As an application, the structure of quasi-socle ideals $I = Q : \mathfrak{m}^q$ $(q \geq 1)$ in a one-dimensional local complete intersection and the question of when the graded rings $\mathrm{G}(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ are Cohen–Macaulay are studied in the case where the ideals I are integral over Q.

1. Introduction and the main results

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let Q be a parameter ideal in A and let $q > 0$ be an integer. We put $I = Q : \mathfrak{m}^q$ and refer to those ideals as quasi-socle ideals in A. In this paper we are interested in the following question about quasi-socle ideals I, which are also the main subject of the researches [1–3].

Question 1.1. (1) Find the conditions under which $I \subseteq \overline{Q}$, where $\overline{Q}$ stands for the integral closure of Q.
(2) When $I \subseteq \overline{Q}$, estimate or describe the reduction number

$$\mathrm{r}_Q(I) = \min\{n \in \mathbb{Z} \mid I^{n+1} = QI^n\}$$

of I with respect to Q in terms of some invariants of Q or A.
(3) Clarify what kind of ring-theoretic properties do the graded rings

$$\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n, \qquad \mathrm{G}(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}, \qquad \text{and} \quad \mathrm{F}(I) = \bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n$$

associated to the ideal I enjoy.

The present research is a continuation of [1–3] and aims mainly at the analysis of the case where A is a complete intersection with $\dim A = 1$. Following Heinzer and Swanson [4], for each parameter ideal Q in a Noetherian local ring $(A, \mathfrak{m})$ we define

$$\mathrm{g}(Q) = \max\{q \in \mathbb{Z} \mid Q : \mathfrak{m}^q \subseteq \overline{Q}\}$$

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and call it the Goto number of Q. In the present paper we are also interested in computing Goto numbers $\mathrm{g}(Q)$ of parameter ideals. In [4] one finds, among many interesting results, that if the base local ring $(A, \mathfrak{m})$ has dimension one, then there exists an integer $k \gg 0$ such that the Goto number $\mathrm{g}(Q)$ is constant for every parameter ideal Q contained in $\mathfrak{m}^k$. We will show that this is not true if $\dim A > 1$, explicitly computing Goto numbers $\mathrm{g}(Q)$ for certain parameter ideals Q in a Noetherian local ring $(A, \mathfrak{m})$ with Gorenstein associated graded ring $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. However, before entering details, let us briefly explain the reasons why we are interested in Goto numbers and quasi-socle ideals as well.

The study of *socle* ideals $Q : \mathfrak{m}$ dates back to the research of Burch [5], where she explored certain socle ideals of finite projective dimension and gave a beautiful characterization of regular local rings (cf. [6, Theorem 1.1]). More recently, Corso and Polini [7,8] studied, with interaction to the linkage theory of ideals, the socle ideals $I = Q : \mathfrak{m}$ of parameter ideals Q in a Cohen–Macaulay local ring $(A, \mathfrak{m})$ and showed that $I^2 = QI$, once A is *not* a regular local ring. Consequently the associated graded ring $\mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ and the fiber cone $\mathrm{F}(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$ are Cohen–Macaulay and so is the ring $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$, if $\dim A \geq 2$. The first author and Sakurai [9–11] explored also the case where the base ring is not necessarily Cohen–Macaulay but Buchsbaum, and showed that the equality $I^2 = QI$ (here $I = Q : \mathfrak{m}$) holds true for numerous parameter ideals Q in a given Buchsbaum local ring $(A, \mathfrak{m})$, whence $\mathrm{G}(I)$ is a Buchsbaum ring, provided that $\dim A \geq 2$ or that $\dim A = 1$ but the multiplicity $\mathrm{e}(A)$ of A is not less than 2. Thus socle ideals $Q : \mathfrak{m}$ still enjoy very good properties even in the case where the base local rings are *not* Cohen–Macaulay.

However a more important fact is the following. If J is an equimultiple Cohen–Macaulay ideal of reduction number one in a Cohen–Macaulay local ring, the associated graded ring $\mathrm{G}(J) = \bigoplus_{n \geq 0} J^n/J^{n+1}$ of J is a Cohen–Macaulay ring and, so is the Rees algebra $\mathcal{R}(J) = \bigoplus_{n \geq 0} J^n$ of J, provided $\mathrm{ht}_A J \geq 2$. One knows the number and degrees of defining equations of $\mathcal{R}(J)$ also, which makes the process of desingularization of $\operatorname{Spec} A$ along the subscheme $\mathrm{V}(J)$ fairly explicit to understand. This observation motivated the ingenious research of Polini and Ulrich [12], where they posed, among many important results, the following conjecture.

Conjecture 1.2 (*[12]*). *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring with $\dim A \geq 2$. Assume that $\dim A \geq 3$ when A is regular. Let $q \geq 2$ be an integer and let Q be a parameter ideal in A such that $Q \subseteq \mathfrak{m}^q$. Then*

$$Q : \mathfrak{m}^q \subseteq \mathfrak{m}^q.$$

This conjecture was settled by Wang [13], whose theorem says:

Theorem 1.3 (*[13]*). *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring with $d = \dim A \geq 2$. Let $q \geq 1$ be an integer and Q a parameter ideal in A. Assume that $Q \subseteq \mathfrak{m}^q$ and put $I = Q : \mathfrak{m}^q$. Then*

$$I \subseteq \mathfrak{m}^q, \qquad \mathfrak{m}^q I = \mathfrak{m}^q Q, \quad \text{and} \quad I^2 = QI,$$

provided that A is not regular if $d \geq 2$ and that $q \geq 2$ if $d \geq 3$.

The research of the first author, Matsuoka, and Takahashi [1] reported a different approach to the Polini–Ulrich conjecture. They proved the following.

Theorem 1.4 (*[1]*). *Let $(A, \mathfrak{m})$ be a Gorenstein local ring with $d = \dim A > 0$ and $\mathrm{e}(A) \geq 3$, where $\mathrm{e}(A)$ denotes the multiplicity of A. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}^2$. Then $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$, $I^3 = QI^2$, and $\mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is a Cohen–Macaulay ring, so that $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$ is also a Cohen–Macaulay ring, provided $d \geq 3$.*

The researches [13,1] are performed independently and their methods of proof are totally different from each other's. The technique of [1] cannot go beyond the restrictions that A is a Gorenstein ring, $q = 2$, and $\mathrm{e}(A) \geq 3$. However, despite these restrictions, the result [1, Theorem 1.1] holds true even in the case where $\dim A = 1$, while Wang's result says nothing about the case where $\dim A = 1$. As is suggested in [1], the one-dimensional case is substantially different from higher-dimensional cases and more complicated to control. This observation has led Goto, Kimura, Matsuoka, and Phuong to the researches [2,3]. In [2] quasi-socle ideals with respect to monomial parameter ideals in Gorenstein numerical semigroup rings over fields are explored. In [3] the authors assumed that $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a Gorenstein ring and parameter ideals Q are diagonal, that is, Q is generated by a system of parameters of the form $x_1^{a_1}, x_2^{a_2}, \ldots, x_d^{a_d}$ $(a_i \geq 1)$, where $x_1, x_2, \ldots, x_d$ are elements of A which generate a reduction of the maximal ideal $\mathfrak{m}$. The present research is a continuation of [1–3] and the main purpose is to go beyond the restriction in [3] that the parameter ideals $Q = (x_1^{a_1}, x_2^{a_2}, \ldots, x_d^{a_d})$ be *diagonal* and the assumption in [2] that the parameter ideals be *monomial*.

To state the main results of the present paper, let us fix some notation. Let A denote a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $\{a_i\}_{1 \leq i \leq d}$ be positive integers and let $\{x_i\}_{1 \leq i \leq d}$ be elements of A with $x_i \in \mathfrak{m}^{a_i}$ for each $1 \leq i \leq d$ such that the initial forms $\{x_i \bmod \mathfrak{m}^{a_i+1}\}_{1 \leq i \leq d}$ constitute a homogeneous system of parameters in $\mathrm{G}(\mathfrak{m})$. Hence $\mathfrak{m}^\ell = \sum_{i=1}^d x_i \mathfrak{m}^{\ell - a_i}$ for $\ell \gg 0$, so that $Q = (x_1, x_2, \ldots, x_d)$ is a parameter ideal in A. Let $q \in \mathbb{Z}$, $I = Q : \mathfrak{m}^q$,

$$\rho = \mathrm{a}(\mathrm{G}(\mathfrak{m}/Q)) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^d a_i, \quad \text{and} \quad \ell = \rho + 1 - q,$$

where $\mathrm{a}(*)$ denote the a-invariants of graded rings ([14, (3.1.4)]). We put

$$\ell_1 = \inf\{n \in \mathbb{Z} \mid \mathfrak{m}^n \subseteq I\} \quad \text{and} \quad \ell_2 = \sup\{n \in \mathbb{Z} \mid I \subseteq Q + \mathfrak{m}^n\}.$$

With this notation our main result is stated as follows.

Theorem 1.5. *Suppose that $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n\ge 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a Cohen–Macaulay ring and consider the following four conditions:*

(1) *$\ell_1 \ge a_i$ for all $1 \le i \le d$.*
(2) *$I \subseteq \overline{Q}$.*
(3) *$\mathfrak{m}^q I = \mathfrak{m}^q Q$.*
(4) *$\ell_2 \ge a_i$ for all $1 \le i \le d$.*

Then one has the implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). If $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring, then one has the equality $I = Q + \mathfrak{m}^\ell$, so that $\ell_1 \le \ell \le \ell_2$, whence conditions (1), (2), (3), and (4) are equivalent to the following :

(5) *$\ell \ge a_i$ for all $1 \le i \le d$.*

Consequently, the Goto number $\mathrm{g}(Q)$ of Q is given by the formula

$$\mathrm{g}(Q) = \left[\mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^{d} a_i + 1\right] - \max\{a_i \mid 1 \le i \le d\},$$

provided $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring ; in particular $\mathrm{g}(Q) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + 1$, if $d = 1$.

Let $R = k[R_1]$ be a homogeneous ring over a field k with $d = \dim R > 0$. We choose a homogeneous system $f_1, f_2, \dots, f_d$ of parameters of R and put $\mathfrak{q} = (f_1, f_2, \dots, f_d)$. Let $M = R_+$. Then, applying Theorem 1.5 to the local ring $A = R_M$, we readily get the following, where $\mathrm{g}(\mathfrak{q}) = \max\{n \in \mathbb{Z} \mid \mathfrak{q} : M^n \text{is integral over } \mathfrak{q}\}$.

Corollary 1.6. *Suppose that R is a Gorenstein ring. Then*

$$\mathrm{g}(\mathfrak{q}) = \left[\mathrm{a}(R) + \sum_{i=1}^{d} \deg f_i + 1\right] - \max\{\deg f_i \mid 1 \le i \le d\}.$$

Hence $\mathrm{g}(\mathfrak{q}) = \mathrm{a}(R) + 1$, if $d = 1$.

Corollary 1.7. *With the same notation as in Theorem 1.5 let $d = 1$ and put $a = a_1$. Assume that $\mathrm{G}(\mathfrak{m})$ is a reduced ring. Then the following conditions are equivalent:*

(1) *$I \subseteq \overline{Q}$.*
(2) *$\mathfrak{m}^q I = \mathfrak{m}^q Q$.*
(3) *$I \subseteq \mathfrak{m}^a$.*
(4) *$\ell_2 \ge a$.*

Later we will give some applications of these results. So, we are now in a position to explain how this paper is organized. Theorem 1.5 will be proven in Section 2. Once we have proven Theorem 1.5, exactly the same technique as developed by [3] works to get a complete answer to Question 1.1 in the case where $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring and Q is a parameter ideal given in Theorem 1.5, which we shall briefly discuss in Section 2.

Sections 3 and 4 are devoted to the analysis of quasi-socle ideals in the ring A of the form $A = B/yB$, where y is part of a system of parameters in a Cohen–Macaulay local ring $(B, \mathfrak{n})$ with $\dim B = 2$. Here we notice that this class of local rings contains all the local complete intersections of dimension one. In Section 3 (resp. Section 4) we focus our attention on the case where B is *not* a regular local ring (resp. B is a regular local ring), and our results are summarized into Theorems 3.1 and 4.1. The proofs given in Sections 3 and 4 are similar to each other and are based on the method developed by Wang [13] in higher-dimensional cases. The techniques are, however, substantially different, depending on the assumptions that B is a regular local ring or not. In Sections 3 and 4 we shall give a careful description of the reason why such a difference should occur.

In what follows, unless otherwise specified, let $(A, \mathfrak{m})$ be a Noetherian local ring with $d = \dim A > 0$. We denote by $\mathrm{e}(A) = \mathrm{e}^0_{\mathfrak{m}}(A)$ the multiplicity of A with respect to the maximal ideal $\mathfrak{m}$. Let $J \subseteq K$ ($\subsetneq A$) be ideals in A. We denote by $\bar{J}$ the integral closure of J. When $K \subseteq \bar{J}$, let

$$\mathrm{r}_J(K) = \min\{n \in \mathbb{Z} \mid K^{n+1} = JK^n\}$$

denote the reduction number of K with respect to J. For each finitely generated A-module M let $\mu_A(M)$ and $\ell_A(M)$ be the number of elements in a minimal system of generators for M and the length of M, respectively. We denote by $\mathrm{v}(A) = \ell_A(\mathfrak{m}/\mathfrak{m}^2)$ the embedding dimension of A.

2. The case where $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring

The purpose of this section is to prove Theorem 1.5. Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $\{a_i\}_{1\le i\le d}$ be positive integers and let $\{x_i\}_{1\le i\le d}$ be elements of A such that $x_i \in \mathfrak{m}^{a_i}$ for each $1 \le i \le d$. Assume that the initial forms $\{x_i \bmod \mathfrak{m}^{a_i+1}\}_{1\le i\le d}$ constitute a homogeneous system of parameters in $\mathrm{G}(\mathfrak{m})$. Let $q \in \mathbb{Z}$ and $Q = (x_1, x_2, \ldots, x_d)$. We put $I = Q : \mathfrak{m}^q$.

Let us begin with the following.

Proposition 2.1. *Let $\ell_3 \in \mathbb{Z}$ and suppose that $\mathfrak{m}^{\ell_3} \subseteq \overline{Q}$. Then $\ell_3 \ge a_i$ for all $1 \le i \le d$.*

Proof. Assume that $\mathfrak{m}^{\ell_3} \subseteq \overline{Q}$ with $\ell_3 \in \mathbb{Z}$. Then $\ell_3 > 0$. We want to show $\ell_3 \ge \max\{a_i \mid 1 \le i \le d\}$. Assume the contrary and let x be an arbitrary element of $\mathfrak{m}$ and put $y = x^{\ell_3}$. Then since y is integral over Q, we have an equation

$$y^n + c_1 y^{n-1} + \cdots + c_n = 0$$

with $n > 0$ and $c_i \in Q^i$ for all $1 \le i \le n$. We put $a = \max\{a_i \mid 1 \le i \le d\}$ (hence $\ell_3 < a$) and let $a = a_u$ with $1 \le u \le d$. Let $B = A/(x_i \mid 1 \le i \le d, i \ne u)$ and $\mathfrak{n} = \mathfrak{m}B$. Let $\overline{f}$ denote, for each $f \in A$, the image of f in B. Then

$$\overline{y}^n + \overline{c_1}\overline{y}^{n-1} + \cdots + \overline{c_n} = 0$$

in B. Therefore, because $i\ell_3 < ia$ and $\overline{c_i} \in Q^iB = x_u^iB \subseteq \mathfrak{n}^{ia}$ (recall that $x_u \in \mathfrak{m}^a$), we get $\overline{c_i} \in \mathfrak{n}^{i\ell_3+1}$ for all $1 \le i \le n$. Consequently, $\overline{c_i}\overline{y}^{n-i} \in \mathfrak{n}^{i\ell_3+1}\mathfrak{n}^{(n-i)\ell_3} = \mathfrak{n}^{n\ell_3+1}$, so that we have $\overline{y}^n = \overline{x^{n\ell_3}} \in \mathfrak{n}^{n\ell_3+1}$. Hence, for every $z \in \mathfrak{n}$, the initial form $z \bmod \mathfrak{n}^2$ of z is nilpotent in the associated graded ring $\mathrm{G}(\mathfrak{n}) = \bigoplus_{n\ge0}\mathfrak{n}^n/\mathfrak{n}^{n+1}$, which is impossible, because $\dim \mathrm{G}(\mathfrak{n}) = \dim B = 1$. Thus $\ell_3 \ge a_i$ for all $1 \le i \le d$. □

We put $\rho = \mathrm{a}(\mathrm{G}(\mathfrak{m}/Q)) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^d a_i$ (cf. [14, (3.1.6)]) and $\ell = \rho + 1 - q$. Let $\ell_1 = \inf\{n \in \mathbb{Z} \mid \mathfrak{m}^n \subseteq I\}$ and $\ell_2 = \sup\{n \in \mathbb{Z} \mid I \subseteq Q + \mathfrak{m}^n\}$.

We are in a position to prove Theorem 1.5.

Proof of Theorem 1.5. (4) ⇒ (3) We may assume $\ell_2 < \infty$. Then, since $I \subseteq Q + \mathfrak{m}^{\ell_2}$, we have $\mathfrak{m}^qI \subseteq \mathfrak{m}^qQ + \mathfrak{m}^{q+\ell_2}$, whence $\mathfrak{m}^qI = \mathfrak{m}^qQ + [Q \cap \mathfrak{m}^{q+\ell_2}]$. Notice that

$$Q \cap \mathfrak{m}^{q+\ell_2} = \sum_{i=1}^d x_i\mathfrak{m}^{q+\ell_2-a_i},$$

because the initial forms $\{x_i \bmod \mathfrak{m}^{a_i+1}\}_{1\le i\le d}$ constitute a homogeneous system of parameters in the Cohen–Macaulay ring $\mathrm{G}(\mathfrak{m})$, and we have $\mathfrak{m}^{q+\ell_2-a_i} \subseteq \mathfrak{m}^q$, since $\ell_2 \ge a_i$ for all $1 \le i \le d$. Thus $\mathfrak{m}^qI = \mathfrak{m}^qQ$.

(3) ⇒ (2) See [15, Section 7, Theorem 2].

(2) ⇒ (1) This follows from Proposition 2.1.

We now assume that $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring. Then $I = Q + \mathfrak{m}^\ell$ by [16] (see [17, Theorem 1.6] also), whence $\ell_1 \le \ell \le \ell_2$, so that the implication (1) ⇒ (4) follows. Therefore, $I \subseteq \overline{Q}$ if and only if $\ell = \rho + 1 - q \ge a_i$ for all $1 \le i \le d$, or equivalently

$$q \le \left[\mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^d a_i + 1\right] - \max\{a_i \mid 1 \le i \le d\}.$$

Thus $\mathrm{g}(Q) = \left[\mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^d a_i + 1\right] - \max\{a_i \mid 1 \le i \le d\}$, so that

$$\mathrm{g}(Q) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + 1,$$

if $d = 1$. □

Remark 2.2 (*cf.* Example 3.7). Unless $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring, the implication (1) ⇒ (4) in Theorem 1.5 does not hold true in general, even when A is a complete intersection and $\mathrm{G}(\mathfrak{m})$ is a Cohen–Macaulay ring. For example, let $V = k[[t]]$ be the formal power series ring over a field k and look at the numerical semigroup ring $A = k[[t^5, t^8, t^{12}]] \subseteq V$. Then $A \cong k[[X, Y, Z]]/(Y^3 - Z^2, X^4 - YZ)$, while $\mathrm{G}(\mathfrak{m}) \cong k[X, Y, Z]/(Y^4, YZ, Z^2)$, whence $\mathrm{G}(\mathfrak{m})$ is a Cohen–Macaulay ring but not a Gorenstein ring. Let $Q = (t^{20})$ in A and let $I = Q : \mathfrak{m}^3$; hence $a_1 = 4$ and $q = 3$. Then $I = (t^{20}, t^{22}, t^{23}, t^{26}, t^{29}) \subseteq \mathfrak{m}^3$ and $I^3 = QI^2$, so that $I \subseteq \overline{Q}$, while $I^2 = QI + (t^{44}) \subseteq Q$ but $t^{44} \notin QI$, since $t^{24} \notin I$. Thus $I^2 = Q \cap I^2 \ne QI$, so that $\mathrm{r}_Q(I) = 2$ and the ring $\mathrm{G}(I)$ is not Cohen–Macaulay. It is direct to check that $\mathfrak{m}^4 \subseteq I$, $\mathfrak{m}^3 \not\subseteq I$, and $I \not\subseteq Q + \mathfrak{m}^4 = \mathfrak{m}^4$ since $t^{22} \in I$ but $t^{22} \notin \mathfrak{m}^4$. Thus $\ell_1 = 4$ and $\ell_2 = 3$.

Proof of Corollary 1.7. Since $Q \subseteq \mathfrak{m}^a$, we readily get the equivalence (3) ⇔ (4). We also have $\overline{\mathfrak{m}^a} = \mathfrak{m}^a$, because the ring $\mathrm{G}(\mathfrak{m})$ is reduced. Hence $\overline{Q} \subseteq \mathfrak{m}^a$. Therefore $I \subseteq \mathfrak{m}^a$, if $I \subseteq \overline{Q}$. Thus all the conditions (1), (2), (3), and (4) are, by Theorem 1.5, equivalent to each other. □

Thanks to Theorem 1.5, similarly as in [3] we have the following complete answer to Question 1.1 for the parameter ideals $Q = (x_1, x_2, \dots, x_d)$. We later need it in the present paper. Let us note a brief proof.

Theorem 2.3. *With the same notation as in Theorem 1.5 assume that $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring. Suppose that $\ell \geq a_i$ for all $1 \leq i \leq d$. Then the following assertions hold true.*

(1) *$\mathrm{G}(I)$ is a Cohen–Macaulay ring, $\mathrm{r}_Q(I) = \lceil \frac{q}{\ell} \rceil$, and $\mathrm{a}(\mathrm{G}(I)) = \lceil \frac{q}{\ell} \rceil - d$, where $\lceil \frac{q}{\ell} \rceil = \min\{n \in \mathbb{Z} \mid \frac{q}{\ell} \leq n\}$.*
(2) *$\mathrm{F}(I)$ is a Cohen–Macaulay ring.*
(3) *$\mathcal{R}(I)$ is a Cohen–Macaulay ring if and only if $q \leq (d-1)\ell$.*
(4) *Suppose that $q > 0$. Then $\mathrm{G}(I)$ is a Gorenstein ring if and only if $\ell \mid q$.*
(5) *Suppose that $q > 0$. Then $\mathcal{R}(I)$ is a Gorenstein ring if and only if $q = (d-2)\ell$.*

To prove Theorem 2.3 we need the following.

Lemma 2.4. *With the same notation as in Theorem 1.5 assume that $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring. If $\ell \geq a_i$ for all $1 \leq i \leq d$, then*

$$Q \cap \mathfrak{m}^{(n+1)\ell+m} \subseteq \mathfrak{m}^m Q I^n$$

for all integers $m, n \geq 0$.

Proof. See Proof of the implication (4) ⇒ (3) in Theorem 1.5 or [3, Proof of Lemma 2.2]. □

Proof of Theorem 2.3. (1) Let $n \geq 0$ be an integer. Then, since $I = Q + \mathfrak{m}^\ell$, we get $I^{n+1} = QI^n + \mathfrak{m}^{(n+1)\ell}$, so that

$$Q \cap I^{n+1} = QI^n + [Q \cap \mathfrak{m}^{(n+1)\ell}] \subseteq QI^n,$$

because $Q \cap \mathfrak{m}^{(n+1)\ell} \subseteq QI^n$ by Lemma 2.4. Therefore $Q \cap I^{n+1} = QI^n$ for all $n \geq 0$, so that $\mathrm{G}(I)$ is a Cohen–Macaulay ring and $\mathrm{r}_Q(I) = \min\{n \in \mathbb{Z} \mid I^{n+1} \subseteq Q\}$. Let $n \in \mathbb{Z}$ and suppose that $I^{n+1} \subseteq Q$. Then $\mathfrak{m}^{(n+1)\ell} \subseteq Q$, whence $(n+1)\ell \geq \rho + 1$ (recall that $\rho = \mathrm{a}(\mathrm{G}(\mathfrak{m}/Q))$). Therefore

$$n + 1 \geq \frac{\rho + 1}{\ell} = \frac{q + \ell}{\ell} = \frac{q}{\ell} + 1,$$

so that $n \geq \frac{q}{\ell}$. Conversely, if $n \geq \frac{q}{\ell}$, then $(n+1)\ell \geq (\frac{q}{\ell} + 1)\ell = q + \ell = \rho + 1$, whence $\mathfrak{m}^{(n+1)\ell} \subseteq Q$, so that $I^{n+1} \subseteq Q$. Thus $\mathrm{r}_Q(I) = \lceil \frac{q}{\ell} \rceil$.

Let Y_i's be the initial forms of x_i's with respect to I. Then $Y_1, Y_2, \dots, Y_d$ is a homogeneous system of parameters of $\mathrm{G}(I)$, whence it constitutes a regular sequence in $\mathrm{G}(I)$. Therefore

$$\mathrm{G}(\bar{I}) \cong \mathrm{G}(I)/(Y_1, Y_2, \dots, Y_d)$$

as graded A-algebras [18], where $\bar{I} = I/Q$. Hence $\mathrm{a}(\mathrm{G}(\bar{I})) = \mathrm{a}(\mathrm{G}(I)) + d$ (cf. [14, (3.1.6)]). Thus $\mathrm{a}(\mathrm{G}(I)) = \lceil \frac{q}{\ell} \rceil - d$, since $\mathrm{a}(\mathrm{G}(\bar{I})) = \mathrm{r}_Q(I) = \lceil \frac{q}{\ell} \rceil$.

(2) By Lemma 2.4

$$\begin{aligned} Q \cap \mathfrak{m}I^{n+1} &= Q \cap [\mathfrak{m}QI^n + \mathfrak{m}^{(n+1)\ell+1}] \\ &= \mathfrak{m}QI^n + [Q \cap \mathfrak{m}^{(n+1)\ell+1}] \\ &\subseteq \mathfrak{m}QI^n. \end{aligned}$$

Hence $Q \cap \mathfrak{m}I^{n+1} = \mathfrak{m}QI^n$ for all $n \geq 0$. Thus $\mathrm{F}(I)$ is a Cohen–Macaulay ring (cf. e.g., [19,20]; recall that $\mathrm{G}(I)$ is a Cohen–Macaulay ring).

(3) The Rees algebra $\mathcal{R}(I)$ of I is a Cohen–Macaulay ring if and only if $\mathrm{G}(I)$ is a Cohen–Macaulay ring and $a(\mathrm{G}(I)) < 0$ ([21, Remark (3.10)], [22]). By assertion (1) the latter condition is equivalent to saying that $\lceil \frac{q}{\ell} \rceil < d$, or equivalently $q \leq (d-1)\ell$.

(4) Notice that $\mathrm{G}(I)$ is a Gorenstein ring if and only if so is the graded ring

$$\mathrm{G}(\bar{I}) = \mathrm{G}(I)/(Y_1, Y_2, \dots, Y_d).$$

Let $r = \mathrm{r}_Q(I) (= \lceil \frac{q}{\ell} \rceil)$. Then $\mathrm{G}(\bar{I})$ is a Gorenstein ring if and only if $(0) : \bar{I}^i = \bar{I}^{r+1-i}$ for all $i \in \mathbb{Z}$ (cf. [17, Theorem 1.6]). Therefore, if $\mathrm{G}(I)$ is a Gorenstein ring, we have $(0) : \bar{I} = \bar{I}^r = \overline{\mathfrak{m}}^{r\ell}$, where $\overline{\mathfrak{m}} = \mathfrak{m}/Q$. On the other hand, since $\bar{I} = \overline{\mathfrak{m}}^\ell$ and $q = \rho + 1 - \ell$, we get

$$(0) : \bar{I} = (0) : \overline{\mathfrak{m}}^\ell = \overline{\mathfrak{m}}^q$$

by [16] (see [17, Theorem 1.6] also). Hence $q = r\ell$, because $\overline{\mathfrak{m}}^{r\ell} = \overline{\mathfrak{m}}^q \neq (0)$ and $q > 0$. Thus $\ell \mid q$ and $r = \frac{q}{\ell}$. Conversely, suppose that $\ell \mid q$; hence $r = \frac{q}{\ell}$. Let $i \in \mathbb{Z}$. Then since $\bar{I} = \overline{\mathfrak{m}}^\ell$, we get $\bar{I}^{r+1-i} = \overline{\mathfrak{m}}^{(r+1-i)\ell}$, while

$$(0) : \bar{I}^i = (0) : \overline{\mathfrak{m}}^{i\ell} = \overline{\mathfrak{m}}^{\rho+1-i\ell}$$

by [17, Theorem 1.6]. Hence $(0) : \bar{I}^i = \bar{I}^{r+1-i}$ for all $i \in \mathbb{Z}$, because

$$(r+1-i)\ell = q + \ell - i\ell = \rho + 1 - i\ell.$$

Thus $\mathrm{G}(\bar{I})$ is a Gorenstein ring, whence so is $\mathrm{G}(I)$.

(5) The Rees algebra $\mathcal{R}(I)$ of I is a Gorenstein ring if and only if $\mathrm{G}(I)$ is a Gorenstein ring and $a(\mathrm{G}(I)) = -2$, provided $d \geq 2$ ([23, Corollary (3.7)]). Suppose that $\mathcal{R}(I)$ is a Gorenstein ring. Then $d \geq 2$ by assertion (2) (recall that $q > 0$). Since $a(\mathrm{G}(I)) = \mathrm{r}_Q(I) - d = -2$, thanks to assertions (1) and (4), we have $\frac{q}{\ell} = \mathrm{r}_Q(I) = d-2$, whence $q = (d-2)\ell$. Conversely, suppose that $q = (d-2)\ell$. Then $d \geq 3$, since $q > 0$. By assertions (1) and (4), $\mathrm{G}(I)$ is a Gorenstein ring with $\mathrm{r}_Q(I) = \frac{q}{\ell} = d-2$, whence $a(\mathrm{G}(I)) = (d-2) - d = -2$. Thus $\mathcal{R}(I)$ is a Gorenstein ring. $\square$

We now discuss Goto numbers. For each Noetherian local ring A let

$$\mathcal{G}(A) = \{\mathrm{g}(Q) \mid Q \text{ is a parameter ideal in } A\}.$$

We explore the value $\min \mathcal{G}(A)$ in the setting of Theorem 1.5 with $\dim A = 1$. For the purpose the following result is fundamental.

Theorem 2.5 (*[4, Theorem 3.1]*). *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension one. Then there exists an integer $k \gg 0$ such that $g(Q) = \min \mathcal{G}(A)$ for every parameter ideal Q of A contained in $\mathfrak{m}^k$.*

Thanks to Theorems 1.5 and 2.5, we then have the following.

Corollary 2.6. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with $\dim A = 1$. Then $\min \mathcal{G}(A) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + 1$, if $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring.*

We close this section with the following.

Proposition 2.7. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring with $\dim A = 1$. Then $\mathrm{v}(A) \leq 2$ if and only if $\min \mathcal{G}(A) = \mathrm{e}(A) - 1$.*

Proof. Suppose that $\mathrm{v}(A) \leq 2$. Then $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(\mathfrak{m})) = \mathrm{e}(A) - 2$. Hence $\min \mathcal{G}(A) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + 1 = \mathrm{e}(A) - 1$ by Corollary 2.6. Conversely, assume that $\min \mathcal{G}(A) = \mathrm{e}(A) - 1$. To prove the assertion, enlarging the field $A/\mathfrak{m}$ if necessary, we may assume that the field $A/\mathfrak{m}$ is infinite (use Theorem 2.5). Let $x \in \mathfrak{m}$ and assume that $Q = (x)$ is a reduction of $\mathfrak{m}$. We put $e = \mathrm{e}(A)$ and $q = \mathrm{g}(Q)$. Then $q \geq e - 1$. Let $B = A/Q$ and $\mathfrak{n} = \mathfrak{m}/Q$. Then $Q : \mathfrak{m}^q \subseteq \overline{Q} \subsetneq A$. Hence $\mathfrak{n}^q \neq (0)$, so that $\mathfrak{n}^i \neq \mathfrak{n}^{i+1}$ for any $0 \leq i \leq q$. Consequently, because $q + 1 \geq e$ and

$$e = \ell_A(A/Q) = \sum_{i \geq 0} \ell_A(\mathfrak{n}^i/\mathfrak{n}^{i+1}) \geq \sum_{i=0}^{q} \ell_A(\mathfrak{n}^i/\mathfrak{n}^{i+1}) \geq q + 1,$$

we get $\mathfrak{n}^{q+1} = (0)$ and $\ell_A(\mathfrak{n}^i/\mathfrak{n}^{i+1}) = 1$ for all $0 \leq i \leq q$. Hence $\ell_A(\mathfrak{n}/\mathfrak{n}^2) \leq 1$, so that $\mathrm{v}(A) \leq 2$. $\square$

3. The case where $A = B/yB$ and B is not a regular local ring

In the following two Sections 3 and 4 we shall restrict our attention on quasi-socle ideals in the ring A of the form $A = B/yB$, where $(B, \mathfrak{n})$ is a Cohen–Macaulay local ring of dimension 2 and y is part of a system of parameters in B. This class of local rings contains all the local complete intersections of dimension one. Typical examples we have in mind are numerical semigroup rings and the main purpose is to go beyond the restriction in [3] that parameter ideals be generated by monomials.

In this section assume that B is *not* a regular local ring; we do not assume that $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a Gorenstein ring. Our goal is the following.

Theorem 3.1. *Let $(B, \mathfrak{n})$ be a Cohen–Macaulay local ring of dimension 2 and assume that B is not a regular local ring. Let n, q be integers such that $n \geq q > 0$. Let $y \in \mathfrak{n}^n$ and assume that y is regular in B. We put $A = B/yB$ and $\mathfrak{m} = \mathfrak{n}/yB$. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}^q$. Then the following assertions hold true, where $m = n - q$.*

(1) *$\mathfrak{m}^q I = \mathfrak{m}^q Q$, $I \subseteq \overline{Q}$, and $Q \cap I^2 = QI$. Hence $\mathrm{g}(Q) \geq n$.*
(2) *$I^2 = QI$, if one of the following conditions is satisfied.*
   (i) *$m \geq q - 1$;*
   (ii) *$m < q - 1$ and $Q \subseteq \mathfrak{m}^{q-m}$;*
   (iii) *$m > 0$ and $Q \subseteq \mathfrak{m}^{q-1}$.*
(3) *Suppose that B is a Gorenstein ring. Then $I^3 = QI^2$ and $\mathrm{G}(I)$ is a Cohen–Macaulay ring, if one of the following conditions is satisfied.*
   (i) *$m < q - 1$ and $Q \subseteq \mathfrak{m}^{q-(m+1)}$;*
   (ii) *$Q \subseteq \mathfrak{m}^{q-1}$.*

We begin with the following.

Lemma 3.2. *Let $(B, \mathfrak{n})$ be a Cohen–Macaulay local ring of dimension 2 and assume that B is not a regular local ring. Let q, ℓ, and m be integers such that $q \geq \ell > 0$ and $m \geq 0$. Let $x \in \mathfrak{n}^\ell$ and $y_i \in \mathfrak{n}(1 \leq i \leq q+m)$ and assume that for all $1 \leq i \leq q+m$, the sequence x, y_i is B-regular. Then we have*

$$\left(x, \prod_{i=1}^{q+m} y_i\right) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{\ell+m}.$$

Proof. Let $\alpha \in (x, \prod_{i=1}^{q+m} y_i) : \mathfrak{n}^q$ and write $\alpha \cdot \prod_{i=1}^{q} y_i = ux + v \cdot \prod_{i=1}^{q+m} y_i$ with $u, v \in B$. Then, since

$$\left(\alpha - v \cdot \prod_{i=q+1}^{q+m} y_i\right) \cdot \prod_{i=1}^{q} y_i \in (x)$$

and since $x, \prod_{i=1}^{q} y_i$ is a B-regular sequence, we get $\alpha - v \cdot \prod_{i=q+1}^{q+m} y_i \in (x)$. Let us write

$$\alpha = wx + v \cdot \prod_{i=q+1}^{q+m} y_i$$

with $w \in B$. We want to show $v \in \mathfrak{n}^\ell$. Let $z \in \mathfrak{n}^\ell$ and write

$$\alpha z \cdot \prod_{i=1}^{q-\ell} y_i = u'x + v' \cdot \prod_{i=1}^{q+m} y_i$$

with $u', v' \in B$. Then, since

$$\alpha z \cdot \prod_{i=1}^{q-\ell} y_i = wxz \cdot \prod_{i=1}^{q-\ell} y_i + vz \cdot \prod_{i=1}^{q-\ell} y_i \cdot \prod_{i=q+1}^{q+m} y_i,$$

we have

$$\left(vz - v' \cdot \prod_{i=q-\ell+1}^{q} y_i\right) \cdot \prod_{i=1}^{q-\ell} y_i \cdot \prod_{i=q+1}^{q+m} y_i \in (x).$$

Therefore, since the sequence $x, \prod_{i=1}^{q-\ell} y_i \cdot \prod_{i=q+1}^{q+m} y_i$ is B-regular, we see $vz \in (x, \prod_{i=q-\ell+1}^{q} y_i)$, so that $v \in (x, \prod_{i=q-\ell+1}^{q} y_i) : \mathfrak{n}^\ell$, because z is an arbitrary element in $\mathfrak{n}^\ell$. We now notice that $\mathfrak{q} = (x, \prod_{i=q-\ell+1}^{q} y_i)$ is a parameter ideal in B such that $\mathfrak{q} \subseteq \mathfrak{n}^\ell$. Then, since B is not a regular local ring, we have $\mathfrak{q} : \mathfrak{n}^\ell \subseteq \mathfrak{n}^\ell$, thanks to [13, Theorem 1.1]. Thus $v \in \mathfrak{n}^\ell$, whence $\alpha \in (x) + \mathfrak{n}^{\ell+m}$. $\square$

Proposition 3.3. *Let $(B, \mathfrak{n})$ be a Cohen–Macaulay local ring of dimension 2 and assume that B is not a regular local ring. Let q, ℓ, and m be integers such that $q \geq \ell > 0$ and $m \geq 0$. Let $x, y \in B$ be a system of parameters of B and assume that $x \in \mathfrak{n}^\ell$ and $y \in \mathfrak{n}^{q+m}$. Then*

(1) $(x, y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{\ell+m}$.
(2) $\mathfrak{n}^q \cdot [(x, y) : \mathfrak{n}^q] \subseteq \mathfrak{n}^q x + (y)$.

Proof. (1) We notice that the ideal $\mathfrak{n}^k$ is, for each integer $k > 0$, generated by the set

$$F_k = \left\{\prod_{i=1}^{k} z_i \mid z_i \in \mathfrak{n} \text{ and } x, z_i \text{ is a system of parameters of } B \text{ for all } 1 \leq i \leq k\right\}.$$

Let $\alpha \in (x, y) : \mathfrak{n}^q$. Let $z \in F_{q+m}$ and $z' \in F_q$ and write

$$z\alpha = ux + vy,$$
$$z'\alpha = u'x + v'y$$

with $u, v, u', v' \in B$. Then $z'z\alpha = z'ux + z'vy = zu'x + zv'y$, whence $y(z'v - zv') \in (x)$, so that $z'v \in (x, z)$, because the sequence x, y is B-regular. Since z' is an arbitrary element of F_k which generates the ideal $\mathfrak{n}^q$, we have

$$v \in (x, z) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{\ell+m}$$

by Lemma 3.2. Hence $z\alpha = ux + vy \in (x) + \mathfrak{n}^{\ell+m} y$, so that

$$\alpha \in [(x) + \mathfrak{n}^{\ell+m} y] : \mathfrak{n}^{q+m},$$

because z is an arbitrary element of F_{q+m}. Since $y \in \mathfrak{n}^{q+m}$, we then have

$$y\alpha = \rho x + \tau y$$

with $\rho \in B$ and $\tau \in \mathfrak{n}^{\ell+m}$. Therefore $\alpha - \tau \in (x)$, so that $\alpha \in (x) + \mathfrak{n}^{\ell+m}$. Thus $(x, y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{\ell+m}$.

(2) The ideal $\mathfrak{n}^q$ is generated by the set

$$F = \{z \in \mathfrak{n}^q \mid y, z \text{ is a system of parameters in } B\}.$$

Let $\alpha \in (x, y) : \mathfrak{n}^q$ and $z, z' \in F$. We write $z\alpha = ux + vy$ and $z'\alpha = u'x + v'y$ with $u, v, u', v' \in B$. We want to show $ux \in \mathfrak{n}^q x$. Since $z'z\alpha = z'ux + z'vy = zu'x + zv'y$, we have $x(z'u - zu') \in (y)$, whence $z'u \in (z, y)$. Therefore $u \in (z, y) : \mathfrak{n}^q$, whence $u \in (z) + \mathfrak{n}^{q+m}$, because $(z, y) : \mathfrak{n}^q \subseteq (z) + \mathfrak{n}^{q+m}$ by assertion (1) (take $x = z$, and $\ell = q$). Thus $ux \in (zx) + \mathfrak{n}^{q+m}x \subseteq \mathfrak{n}^q x$, whence $\mathfrak{n}^q \cdot [(x, y) : \mathfrak{n}^q] \subseteq \mathfrak{n}^q x + (y)$. $\square$

We need also the following result to prove Theorem 3.1.

Proposition 3.4. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring with $d = \dim A > 0$. Let Q be a parameter ideal in A and $q > 0$ an integer. We put $I = Q : \mathfrak{m}^q$. Then $I^3 = QI^2$ and $\mathrm{G}(I)$ is a Cohen–Macaulay ring, if $I \subseteq Q + \mathfrak{m}^{q-1}$ and $\mathfrak{m}^q I = \mathfrak{m}^q Q$.*

Proof. We have $\mathfrak{m}^q I^i = \mathfrak{m}^q Q^i$ and $Q^i \cap I^{i+1} = Q^i I$ for all $i \geq 1$ (cf. [1, Corollary 2.3]). Therefore, since $Q \cap I^2 = QI$, we may assume that $I^2 \not\subseteq Q$. Notice that $\mathfrak{m}I^2 = \mathfrak{m}I \cdot I \subseteq (Q + \mathfrak{m}^q) \cdot I \subseteq Q$ and we have $I^2 \subseteq Q : \mathfrak{m}$. Hence $Q : \mathfrak{m} = Q + I^2$, because A is a Gorenstein ring. We similarly have $\mathfrak{m}I^3 \subseteq \mathfrak{m}I \cdot I^2 \subseteq (\mathfrak{m}Q + \mathfrak{m}^q) \cdot I^2 = \mathfrak{m}I^2 \cdot Q + \mathfrak{m}^q I^2 \subseteq Q^2$, so that $I^3 \subseteq Q^2 : \mathfrak{m} = Q \cdot [Q : \mathfrak{m}] = Q^2 + QI^2$. Therefore $I^3 = [Q^2 + QI^2] \cap I^3 = [Q^2 \cap I^3] + QI^2 = Q^2 I + QI^2 = QI^2$. Hence $I^3 = QI^2$, which implies, because $Q \cap I^2 = QI$, that $\mathrm{G}(I)$ is a Cohen–Macaulay ring. $\square$

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $Q = (\overline{x})$ with $x \in \mathfrak{n}$, where $\overline{x}$ denotes the image of x in A. We put $J = (x, y) : \mathfrak{n}^q$; hence $I = JA$. We have by Proposition 3.3 that $J \subseteq (x) + \mathfrak{n}^{m+1}$ and $\mathfrak{n}^q J \subseteq \mathfrak{n}^q x + (y)$ (take $\ell = 1$). Hence $\mathfrak{m}^q I = \mathfrak{m}^q Q$, so that $I \subseteq \overline{Q}$ (cf. [15]). Let $\alpha \in Q \cap I^2$ and write $\alpha = \overline{x}\beta$ with $\beta \in A$. Then, for all $\gamma \in \mathfrak{m}^q$, we have $\alpha\gamma = \overline{x} \cdot \beta\gamma \in \mathfrak{m}^q I^2 \subseteq Q^2 = (\overline{x}^2)$, so that $\beta\gamma \in (\overline{x}) = Q$. Therefore $\beta \in Q : \mathfrak{m}^q = I$, whence $\alpha = \overline{x}\beta \in QI$. Thus $Q \cap I^2 = QI$, which proves assertion (1).

If $m \geq q - 1$, we have $J \subseteq (x) + \mathfrak{n}^{m+1} \subseteq (x) + \mathfrak{n}^q$, whence $I \subseteq Q + \mathfrak{m}^q$. Therefore $I^2 \subseteq Q$, so that $I^2 = QI$ by assertion (1). Suppose that $m < q - 1$ and $Q \subseteq \mathfrak{m}^{q-m}$. We choose the element x so that $x \in \mathfrak{n}^{q-m}$. Then, taking $\ell = q - m$, by Proposition 3.3 (1) we get $J = (x, y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^q$. Hence $I \subseteq Q + \mathfrak{m}^q$. Thus $I^2 = QI$. Suppose now that $m > 0$ and $Q \subseteq \mathfrak{m}^{q-1}$. To show $I^2 = QI$, we may assume by condition (ii) that $m < q - 1$. Then $Q \subseteq \mathfrak{m}^{q-m}$, since $Q \subseteq \mathfrak{m}^{q-1}$ and $m > 0$. Hence $I^2 = QI$. This proves assertion (2).

Let us consider assertion (3). Suppose that B is a Gorenstein ring and assume that condition (i) is satisfied. We choose the element x so that $x \in \mathfrak{n}^{q-(m+1)}$. Then $J = (x, y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{q-1}$ (take $\ell = q - (m + 1)$), whence $I \subseteq Q + \mathfrak{m}^{q-1}$, so that the result follows from Proposition 3.4. Assume that condition (ii) is satisfied. By assertion (2) we may assume that $m < q - 1$. Then, since $\mathfrak{m}^{q-1} \subseteq \mathfrak{m}^{q-(m+1)}$, we have $Q \subseteq \mathfrak{m}^{q-(m+1)}$, so that condition (i) is satisfied, whence the result follows. This completes the proof of Theorem 3.1. $\square$

Let us note here some concrete examples. Let $n \geq 0$ be an integer and put $a = 6n + 5$, $b = 6n + 8$, and $c = 9n + 12$. Then $0 < a < b < c$ and $\mathrm{GCD}(a, b, c) = 1$. Let $A = k[[t^a, t^b, t^c]] \subseteq k[[t]]$, where $k[[t]]$ denotes the formal power series ring over a field k. Then

$$A \cong k[[X, Y, Z]]/(Y^3 - Z^2, X^{3n+4} - Y^{3n+1}Z),$$

where $k[[X, Y, Z]]$ denotes the formal powers series ring. Let $\mathfrak{m}$ be the maximal ideal in A. Then

$$\mathrm{G}(\mathfrak{m}) \cong k[X, Y, Z]/(Y^{3n+4}, Y^{3n+1}Z, Z^2).$$

Hence A is a complete intersection with $\dim A = 1$, whose associated graded ring $\mathrm{G}(\mathfrak{m})$ is not a Gorenstein ring but Cohen–Macaulay. We put

$$B = k[[X, Y, Z]]/(Y^3 - Z^2)$$

and let y denote the image of $X^{3n+4} - Y^{3n+1}Z$ in B. Let $\mathfrak{n} = (X, Y, Z)B$ be the maximal ideal in B. Then B is not a regular local ring and $A = B/yB$. We have $y \in \mathfrak{n}^{3n+2}$ and y is part of a system of parameters of B. Therefore by Theorem 3.1 (1), (2), and (3) we have the following.

Example 3.5. Let $0 < q \leq 3n+2$ be an integer and put $m = (3n+2) - q$. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}^q$. Then the following assertions hold true.

(1) $\mathfrak{m}^q I = \mathfrak{m}^q Q$, $I \subseteq \overline{Q}$, and $Q \cap I^2 = QI$. Hence $\mathrm{g}(Q) \geq 3n + 2$.
(2) $I^2 = QI$, if one of the following conditions is satisfied.
  (i) $m \geq q - 1$;
  (ii) $m < q - 1$ and $Q \subseteq \mathfrak{m}^{q-m}$;
  (iii) $m > 0$ and $Q \subseteq \mathfrak{m}^{q-1}$.
(3) $I^3 = QI^2$ and the ring $\mathrm{G}(I)$ is Cohen–Macaulay, if one of the following conditions is satisfied.
  (i) $m < q - 1$ and $Q \subseteq \mathfrak{m}^{q-(m+1)}$;
  (ii) $Q \subseteq \mathfrak{m}^{q-1}$.

Remark 3.6. In Example 3.5 (3) the equality $I^2 = QI$ does not necessarily hold true. For example, let $n = 0$; hence $A = k[[t^5, t^8, t^{12}]]$. Let $Q = (t^5)$ in A and $I = Q : \mathfrak{m}^2$. Then $I = (t^5, t^{12}, t^{16}) \subseteq \overline{Q}$ and $\mathrm{r}_Q(I) = 2$.

As we see in the following examples, the assumption that $y \in \mathfrak{n}^q$ in Theorem 3.1 is crucial in order to control Cohen–Macaulayness in $\mathrm{G}(I)$ for quasi-socle ideals $I = Q : \mathfrak{m}^q$.

Example 3.7. In Example 3.5 take $n = 0$ and look at the local ring $A = k[[t^5, t^8, t^{12}]]$. Hence

$$A \cong k[[X, Y, Z]]/(Y^3 - Z^2, X^4 - YZ).$$

Let $0 < s \in \langle 5, 8, 12\rangle := \{5\alpha + 8\beta + 12\gamma \mid 0 \leq \alpha, \beta, \gamma \in \mathbb{Z}\}$ and $Q = (t^s)$ in A, *monomial* parameters. Let us consider the quasi-socle ideal $I = Q : \mathfrak{m}^3$. Then we always have $I \subseteq \overline{Q}$, but $\mathrm{G}(I)$ is Cohen–Macaulay (resp. the equality $\mathfrak{m}^3 I = \mathfrak{m}^3 Q$ holds true) if and only if $s \in \{5, 10, 12, 15, 17\}$ (resp. $s \in \{5, 12, 17\}$), or equivalently $Q \cap I^2 = QI$. Thus Cohen–Macaulayness in $\mathrm{G}(I)$ is rather wild, as we summarize in the following table.

s	I	$\mathfrak{m}^3 I = \mathfrak{m}^3 Q$	$\mathrm{G}(I)$ is CM	$\mathrm{r}_Q(I)$
5	$\mathfrak{m} = (t^5, t^8, t^{12})$	Yes	Yes	3
8	(t^8, t^{10}, t^{17})	No	No	3
10	$(t^{10}, t^{12}, t^{13}, t^{16})$	No	Yes	2
12	$(t^{12}, t^{15}, t^{18}, t^{21})$	Yes	Yes	1
13	$(t^{13}, t^{15}, t^{16}, t^{22})$	No	No	2
15	$(t^{15}, t^{17}, t^{18}, t^{21}, t^{24})$	No	Yes	2
16	$(t^{16}, t^{18}, t^{22}, t^{25})$	No	No	2
17	$(t^{17}, t^{20}, t^{23}, t^{24}, t^{26})$	Yes	Yes	1
18	$(t^{18}, t^{20}, t^{21}, t^{24}, t^{27})$	No	No	2
≥ 20	$(t^s, t^{s+2}, t^{s+3}, t^{s+6}, t^{s+9})$	No	No	2

4. The case where $A = B/yB$ and B is a regular local ring

In this section let us assume that $(B, \mathfrak{n})$ is a regular local ring of dimension 2 and $A = B/yB$, where y is part of a system of parameters in B. Hence $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a Gorenstein ring, so that the basic assumption in Theorem 1.5 is satisfied. Recall that $\mathrm{v}(A) \leq 2$ and $\min \mathcal{g}_s(A) = \mathrm{e}(A) - 1$ (Proposition 2.7).

Our goal of this time is the following.

Theorem 4.1. *Let $(B, \mathfrak{n})$ be a regular local ring of dimension 2. Let n, q be integers such that $n > q > 0$ and put $m = n - q$. Let $0 \neq y \in \mathfrak{n}^n$ and put $A = B/yB$ and $\mathfrak{m} = \mathfrak{n}/yB$. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}^q$. Then the following assertions hold true.*

(1) $\mathfrak{m}^q I = \mathfrak{m}^q Q$, $I \subseteq \overline{Q}$, *and* $Q \cap I^2 = QI$.
(2) $I^2 = QI$, *if one of the following conditions is satisfied.*
 (i) $m \geq q$;
 (ii) $m < q$ *and* $Q \subseteq \mathfrak{m}^{q-(m-1)}$.
(3) $I^3 = QI^2$ *and the ring* $\mathrm{G}(I)$ *is Cohen–Macaulay, if one of the following conditions is satisfied.*
 (i) $m < q$ *and* $Q \subseteq \mathfrak{m}^{q-m}$;
 (ii) $Q \subseteq \mathfrak{m}^{q-1}$.

Our proof of Theorem 4.1 is, this time, based on the following.

Proposition 4.2. *Let $(B, \mathfrak{n})$ be a regular local ring of dimension 2 and let x, y be a system of parameters of B. Let $q, \ell > 0$ and $m \geq 0$ be integers such that $q + 1 \geq \ell$ and assume that $x \in \mathfrak{n}^\ell$ and $y \in \mathfrak{n}^{q+m}$. Then the following assertions hold true.*

(1) $(x, y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{\ell+m-1}$.
(2) *Suppose that $m > 0$. Then* $\mathfrak{n}^q \cdot [(x, y) : \mathfrak{n}^q] \subseteq \mathfrak{n}^q x + (y)$.

Proof. (1) Enlarging the field $B/\mathfrak{n}$ if necessary, we may assume that the field $B/\mathfrak{n}$ is infinite. Let $\mathrm{G}(\mathfrak{n}) = \bigoplus_{n \geq 0} \mathfrak{n}^n/\mathfrak{n}^{n+1}$ denote the associated graded ring of B. Then $\mathrm{G}(\mathfrak{n})$ is the polynomial ring with two indeterminates over $B/\mathfrak{n}$. For each element $0 \neq f \in B$ let $\mathrm{o}_\mathfrak{n}(f) = \max\{n \in \mathbb{Z} \mid y \in \mathfrak{n}^n\}$ and let $f^* = f \bmod \mathfrak{n}^{\mathrm{o}_\mathfrak{n}(f)+1}$ be the initial form of f; hence f^* is $\mathrm{G}(\mathfrak{n})$-regular. For each integer $k > 0$, the ideal $\mathfrak{n}^k$ is generated by the set

$$F_k = \{z \in \mathfrak{n}^k \mid z \in \mathfrak{n}^k \setminus \mathfrak{n}^{k+1} \text{ and } x^*,\ z^* \text{ is a homogeneous system of parameters in } \mathrm{G}(\mathfrak{n})\}.$$

Now let $\alpha \in (x, y) : \mathfrak{n}^q$, $z \in F_{q+m}$, and $z' \in F_q$. Then $z\alpha = ux + vy$ and $z'\alpha = u'x + v'y$ for some $u, v, u', v' \in B$. Hence, because the sequence x, y is B-regular, comparing two expressions of $z'z\alpha$, we get $z'v \in (x, z)$, whence $v \in (x, z) : \mathfrak{n}^q$. Recall now that $(x, z) : \mathfrak{n}^q = (x, z) + \mathfrak{n}^{\ell'}$ with

$$\begin{aligned}\ell' &= [\mathrm{a}(\mathrm{G}(\mathfrak{n}/(x, z))) + 1] - q \\ &= \left[\mathrm{a}(\mathrm{G}(\mathfrak{n})/(x^*, z^*)) + 1\right] - q \\ &= [\mathrm{a}(\mathrm{G}(\mathfrak{n})) + \mathrm{o}_{\mathfrak{n}}(x) + \mathrm{o}_{\mathfrak{n}}(z) + 1] - q \\ &\geq [(-2) + \ell + (q + m) + 1] - q = \ell + m - 1\end{aligned}$$

(cf. [16]; see [17, Theorem 1.6] also), where $\mathrm{a}(*)$ denotes the a-invariant of the corresponding graded ring ([14, (3.1.4)]). Therefore

$$z\alpha = ux + vy \in (x) + (zy) + \mathfrak{n}^{\ell'}y \subseteq (x) + \mathfrak{n}^{\ell+m-1}y,$$

because $\ell' \geq \ell + m - 1$ and $z \in \mathfrak{n}^{q+m}$ with $q \geq \ell - 1$. Hence $\alpha \in [(x) + \mathfrak{n}^{\ell+m-1}y] : \mathfrak{n}^{q+m}$, so that $\alpha y \in (x) + \mathfrak{n}^{\ell+m-1}y$, whence $\alpha \in (x) + \mathfrak{n}^{\ell+m-1}$, since the sequence x, y is B-regular. Thus $(x, y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{\ell+m-1}$.

(2) The ideal $\mathfrak{n}^q$ is generated by the set $F = \{z \in \mathfrak{n}^q \mid y, z \text{ is a } B\text{-regular sequence}\}$. Let $\alpha \in (x, y) : \mathfrak{n}^q$ and $z, z' \in F$. Then $z\alpha = ux + vy$ and $z'\alpha = u'x + v'y$ for some $u, v, u', v' \in B$. We want to show that $z\alpha \in \mathfrak{n}^q x + (y)$. Because the sequence y, x is B-regular, comparing two expressions of $z'z\alpha$, we get $z'u \in (z, y)$, whence $u \in (z, y) : \mathfrak{n}^q$. Notice now that $(z, y) : \mathfrak{n}^q \subseteq (z) + \mathfrak{n}^{q+m-1}$ by assertion (1) (take $x = z$ and $q = \ell$). Then

$$z\alpha = ux + vy \in (zx) + \mathfrak{n}^{q+m-1}x + (y) \subseteq \mathfrak{n}^q x + (y),$$

since $m > 0$, whence we have $\mathfrak{n}^q \cdot [(x, y) : \mathfrak{n}^q] \subseteq \mathfrak{n}^q x + (y)$. $\square$

Our proof of Theorem 4.1 is now similar to that of Theorem 3.1. We briefly note it.

Proof of Theorem 4.1. Let $Q = (\overline{x})$ with $x \in \mathfrak{n}$, where $\overline{x}$ denotes the image of x in A. Let $J = (x, y) : \mathfrak{n}^q$. Then by Proposition 4.2 that $J \subseteq (x) + \mathfrak{n}^m$ and $\mathfrak{n}^q J \subseteq \mathfrak{n}^q x + (y)$ (take $\ell = 1$). Hence $\mathfrak{m}^q I = \mathfrak{m}^q Q$, so that $I \subseteq \overline{Q}$. We have $Q \cap I^2 = QI$ exactly for the same reason as is in Proof of Theorem 3.1.

To see assertion (2), suppose that $m \geq q$. Then $J \subseteq (x) + \mathfrak{n}^q$, whence $I \subseteq Q + \mathfrak{m}^q$. Therefore $I^2 \subseteq Q$, so that $I^2 = QI$ by assertion (1). Suppose that $m < q - 1$ and $Q \subseteq \mathfrak{m}^{q-m+1}$. We choose the element x so that $x \in \mathfrak{n}^{q-m+1}$. Then, taking $\ell = q - m + 1$, by Proposition 4.2 (1) we get $J = (x, y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^q$. Hence $I \subseteq Q + \mathfrak{m}^q$, so that $I^2 \subseteq Q$, whence $I^2 = QI$.

Suppose that condition (i) in assertion (3) is satisfied. We choose the element x so that $x \in \mathfrak{n}^{q-m}$. Then $J = (x, y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{q-1}$ (take $\ell = q - m$), whence $I \subseteq Q + \mathfrak{m}^{q-1}$, so that the result follows from Proposition 3.4. Suppose that condition (ii) in assertion (3) is satisfied but $m < q$. Then $Q \subseteq \mathfrak{m}^{q-m}$, since $Q \subseteq \mathfrak{m}^{q-1}$ and $m > 0$. Hence the result follows. $\square$

Let us give a consequence of Theorems 2.3 and 4.1.

Corollary 4.3. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring with $\dim A = 1$ and $\mathrm{v}(A) = 2$. Let $q > 0$ be an integer such that $\mathrm{e}(A) > q > 0$ and put $m = \mathrm{e}(A) - q$. Then if $m \geq q - 2$, for every parameter ideal Q in A the following assertions hold true, where $I = Q : \mathfrak{m}^q$.*

(1) $\mathfrak{m}^q I = \mathfrak{m}^q Q$ *and* $\mathrm{r}_Q(I) \leq 3$.
(2) $q = 3$ *and Q is a reduction of* $\mathfrak{m}$, *if* $\mathrm{r}_Q(I) = 3$.
(3) $\mathrm{G}(I)$ *is a Cohen–Macaulay ring.*

Proof. Let $e = \mathrm{e}(A)$. Passing to the $\mathfrak{m}$-adic completion of A, we may assume that $A = B/yB$, where $(B, \mathfrak{n})$ is a regular local ring of dimension 2 and $0 \neq y \in \mathfrak{n}^e$. Hence $\mathfrak{m}^q I = \mathfrak{m}^q Q$ by Theorem 4.1 (1). We must show that $\mathrm{r}_Q(I) \leq 3$ and $\mathrm{G}(I)$ is a Cohen–Macaulay ring. Thanks to Theorem 4.1 (2), we may assume $m < q$ and $Q \not\subseteq \mathfrak{m}^{q-m}$. Hence $m = q - 2$ or $m = q - 1$. Let $Q = (\overline{x})$ with $x \in \mathfrak{n}$, where $\overline{*}$ denotes the image in A. Then $q - m \neq 1$ since $x \notin \mathfrak{n}^{q-m}$, whence $m = q - 2$, that is $e = 2q - 2$. Let $\mathfrak{n} = (x, z)$ with $z \in B$ and let $D = B/xB$. Then D is a DVR. Let us write $yD = z^\ell D$ with $\ell \geq e > q$ and we have $(x, y) : \mathfrak{n}^q = (x) + \mathfrak{n}^{\ell-q}$. If $\ell > e$, then $I = Q + \mathfrak{m}^{\ell-q} \subseteq Q + \mathfrak{m}^{e+1-q} = Q + \mathfrak{m}^{q-1}$, so that $I^2 = QI$ by Proposition 3.4. Assume that $\ell = e$. Then x^*, y^* is a homogeneous system of parameters in $\mathrm{G}(\mathfrak{n})$ with $\deg x^* = 1$ and $\deg y^* = e$, so that Q is a reduction of $\mathfrak{m}$ and $I = Q + \mathfrak{m}^{\ell'}$ by [16], where

$$\begin{aligned}\ell' &= \mathrm{a}(\mathrm{G}(\mathfrak{m}/Q)) + 1 - q \\ &= \left[\mathrm{a}(\mathrm{G}(\mathfrak{n})/(x^*, y^*)) + 1\right] - q \\ &= [(-2) + (1 + e)] + 1 - q \\ &= e - q \\ &= m.\end{aligned}$$

Therefore $\mathrm{r}_Q(I) = \lceil \frac{q}{m} \rceil = \lceil \frac{q}{q-2} \rceil$, thanks to Theorem 2.3 (1). Hence, if $\mathrm{r}_Q(I) \geq 4$, then $\frac{q}{q-2} > 3$, so that $q < 3$. This is impossible, since $m = q - 2 > 0$. Thus $\mathrm{r}_Q(I) \leq 3$. We similarly have $q = 3$, if $\mathrm{r}_Q(I) = 3$. $\square$

Let $4 \leq a < b$ be integers such that $\mathrm{GCD}(a, b) = 1$ and let

$$H = \langle a, b \rangle := \{a\alpha + b\beta \mid 0 \leq \alpha, \beta \in \mathbb{Z}\}$$

be the numerical semigroup generated by a, b. Let $A = k[[t^a, t^b]] (\subseteq k[[t]])$ be the numerical semigroup ring of H and $\mathfrak{m} = (t^a, t^b)$ the maximal ideal in A, where $k[[t]]$ is the formal power series ring over a field k. Then

$$A \cong k[[X, Y]]/(X^b - Y^a),$$

where $B = k[[X, Y]]$ denotes the formal power series ring. Hence, applying Corollaries 2.6 and 4.3, we get the following.

Corollary 4.4. *The following assertions hold true.*

(1) $\min \mathcal{g}(A) = a - 1 \geq 3$.
(2) *Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}^3$. Then $I^4 = QI^3$ and $\mathrm{G}(I)$ is a Cohen–Macaulay ring.*

Remark 4.5. To see that the results of Theorem 4.1 are sharp, the reader may consult [2,3] for examples of monomial parameter ideals $Q = (t^s)$ $(0 < s \in H)$ in numerical semigroup rings $A = k[[H]]$. See [3, Proposition 10] for the case where $H = \langle a, b \rangle$ with $\mathrm{GCD}(a, b) = 1$. Let us pick up the simplest ones.

(1) The equality $I^2 = QI$ does not necessarily hold true. Let $A = k[[t^3, t^4]]$, $Q = (t^3)$, and $I = Q : \mathfrak{m}^2$. Then $I = \mathfrak{m} \subseteq \overline{Q}$ and $\mathrm{r}_Q(I) = 2$.
(2) The reduction number $\mathrm{r}_Q(I)$ could be not less than 3. Let $A = k[[t^4, t^5]]$, $Q = (t^4)$, and $I = Q : \mathfrak{m}^3$. Then $I = \mathfrak{m} \subseteq \overline{Q}$ and $\mathrm{r}_Q(I) = 3$.
(3) The ring $\mathrm{G}(I)$ is not necessarily Cohen–Macaulay. Let $A = k[[t^5, t^6]]$, $Q = (t^{11})$, and $I = Q : \mathfrak{m}^4$. Then $I = (t^{11}, t^{12}, t^{15}) \subseteq \overline{Q}$ and $\mathrm{r}_Q(I) = 3$. However, since $t^{36} \in Q \cap I^3$ but $t^{36} \notin QI^2$, we have $Q \cap I^3 \neq QI^2$, so that $\mathrm{G}(I)$ is not a Cohen–Macaulay ring.

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